Algebraic entropy of elementary amenable groups.

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Abstract

We prove that any finitely generated elementary amenable group of zero (algebraic) entropy contains a nilpotent subgroup of finite index or, equivalently, any finitely generated elementary amenable group of exponential growth is of uniformly exponential growth. We also show that 0 is an accumulation point of the set of entropies of elementary amenable groups.

1 Introduction.

Let G be a group generated by a finite set X. As usual, we denote by $||g||_X$ the *word length* of an element $g \in G$ with respect to X, i.e., the length of a shortest word over the alphabet $X \cup X^{-1}$ which represents g.

In this paper we study the *growth function* $\gamma_G^X : \mathbb{N} \longrightarrow \mathbb{N}$ of G which is defined by

$$\gamma_G^X(n) = card\ \{g \in G\ :\ ||g||_X \leq n\}.$$

Originally growth considerations in group theory were introduced in 50-th by Efremovich [13], Švarc [40], and Følner [15], and (independently) in 60-th by Milnor [29] with motivations from differential geometry and theory of invariant means.

The *exponential growth rate* of G with respect to X is the number

$$\omega(G, X) = \lim_{n\to\infty} \sqrt[n]{\gamma_G^X(n)}.$$

The above limit exists by submultiplicativity of γ_G^X [42, Theorem 4.9]. The quantity

$$\omega(G) = \inf_X \omega(G, X)$$

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is called a *minimal exponential growth rate* of G (the infimum is taken over all finite generating sets of G). Finally, the *(algebraic) entropy* of the group G is defined by the formula

$$h(G) = \log \omega(G).$$

This notion of entropy comes from geometry and should not be confused with the notion of entropy for a pair (G, μ), where μ is a symmetric probability measure on a group G, as defined in [4]. In particular, if G is a fundamental group of a compact Riemannian manifold of unit diameter, then $h(G)$ is a lower bound for the topological entropy of the geodesic flow of the manifold [28]. The exponential growth rates appear also in the study of random walks on the Cayley graphs of finitely generated groups. Details and backgrounds can be found in [18], [24].

The group G is said to be of *exponential growth* if $\omega(G, X) > 1$ and of *subexponential growth* if $\omega(G, X) = 0$. If there exist constants $C, d > 0$ such that $\gamma_G^X(n) \leq Cn^d$ for all $n \in \mathbb{N}$, then G is said to be *of polynomial growth.* These definitions depend on the group G only, not on the choice of finite generating sets. We refer to [10], [16], [30], [41], and [44], for classical results concerning growth of various classes of groups. Further, one says that G has *uniformly exponential growth* if $\omega(G) > 1$ (or, equivalently, $h(G) > 0$). The following important problem goes back to the book [22] and can be found in [17] as well as in [18] and [24].

Question 1.1. *Does there exist a finitely generated group of non–uniform exponential growth, i.e., of exponential growth and of zero entropy?*

On one hand, the affirmative answer was recently obtained by J. Wilson [43]. On the other hand, there are many examples of classes of groups which are known to have uniformly exponential growth. Let us mention some of them.

• Hyperbolic groups containing no cyclic subgroups of finite index [27].

• Free products with amalgamations $G *_{A=B} H$ satisfying the condition $(|G : A| - 1)(|H : B| - 1) \geq 2$ and HNN–extensions $G*_A$ associated with a monomorphism $\phi_1, \phi_2 : A \to G$, where $|G : \phi_1(A)| + |G : \phi_2(A)| \geq 3$ (see [9]).

• One–relator groups of exponential growth [20].

• Solvable groups of exponential growth [32] (the particular case of polycyclic groups was considered independently in [3]).

• Linear groups of exponential growth [14].

The main goal of the present paper is to investigate the case of elementary amenable groups and discuss certain applications of the obtained results.

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2 Main results

In order to explain the Hausdorff–Banach–Tarski paradox, von Neumann [31] introduced the class of amenable groups. He showed that all finite and abelian groups are amenable and the class of amenable groups, AG, is closed under four standard operations of constructing new groups from given ones:

(S) Taking subgroups.

(Q) Taking quotient groups.

(E) Group extensions.

(U) Direct limits (that is, for a given set of groups $\{G_\lambda\}_{\lambda\in\Lambda}$ such that for any $\lambda, \mu \in \Lambda$, there is $\nu \in \Lambda$ satisfying $G_\lambda \cup G_\mu \subseteq G_\nu$, one takes $\bigcup\limits_{\lambda\in\Lambda} G_\lambda$).

As in [12], let EG be the class of *elementary amenable groups* that is the smallest class which contains all abelian and finite groups, and closed under the operations (S)–(U). In particular, EG contains all solvable groups. However, it is easy to construct a finitely generated group $G \in EG$ that is not even virtually solvable. (Recall that a group G is a virtually $\mathcal{P}$ group, where P is a class of groups, if there exists a subgroup H of finite index in G such that $H \in \mathcal{P}$.)

The main result of this paper is the following.

Theorem 2.1. *Let G be a finitely generated elementary amenable group of zero entropy. Then G contains a nilpotent subgroup of finite index. In particular, any elementary amenable group of exponential growth is of uniformly exponential growth.*

This extends the result of Chou, saying that any elementary amenable group of subexponential growth contains a nilpotent subgroup of finite index, as well as the result of the author from [32], where the analog of Theorem 2.1 was proved in the case of solvable groups.

In [35], Rosset proved that if G is a group of subexponential growth, H is a normal subgroup of G, and G/H is solvable, then H is finitely generated. The methods developed in the present paper allows to obtain the following more general result on the structure of normal subgroups of groups with zero entropy.

Theorem 2.2. *Let G be a finitely generated group of zero entropy and let H be a subgroup of G such that the quotient group G/H is elementary amenable. Then H is finitely generated.*

In particular, Theorem 2.2 provides a natural approach to prove that a group has uniform exponential growth.

Given a closed manifold M endowed with a Riemannian metric g, we denote by $h_{top}(M, g)$ the topological entropy of the geodesic flow on M (precise definitions and certain properties can be found in [28], [1]). The connection between the topological entropy of geodesic flows and homotopic properties of

Riemannian manifold was observed by Dinamburg in [11], where it was proved that $h_{top}(M,g) > 0$ whenever $\pi_1(M)$ has exponential growth. It is an interesting question to describe the behaviour of $h_{top}(M,g)$ for a given M, when the metric varies. One can show that $h_{top}(M,g)$ can be made arbitrary large by local variations of the metric. Thus the question above is the question of the existence of a non–trivial lower bound for the quantity

$$h(M) = \inf_{Vol(M,g)=1} h_{top}(M,g), \tag{1}$$

where the infimum is taken over all Riemannian metrics on M normalized by the condition $Vol(M,g) = 1$.

It is known that $h(M) > 0$ for any manifold admitting negative sectional curvature [26], [8]. The proof of this result given in [8] is of purely geometric nature. Here we illustrate another (algebraic) approach to the estimation of $h(M)$ from below. According to the main result of [28], we have

$$h(M) \geq h(\pi_1(M))$$

for every closed manifold M. Therefore, Theorem 2.1 yields the following generalization of the Dinamburg theorem in the particular case of manifolds with elementary amenable fundamental groups.

Corollary 2.1. *Let M be a closed Riemannian manifold and $h(M)$ denote the topological entropy of M defined by (1). Then $h(M) > 0$ whenever the fundamental group $\pi_1(M)$ is elementary amenable and has exponential growth.*

We note that Corollary 2.1 provides purely topological conditions for the positivity of the entropy. However, the inequality $h(M) > 0$ can be true even for simply–connected manifolds (see [5]).

In conclusion we discuss a question concerning possible values of the quantity $\omega(G)$, where G is a finitely generated group. For a class $\mathcal{U}$ of finitely generated groups, we set

$$\Omega(\mathcal{U}) = \{\omega(G) \ : \ G \in \mathcal{U}\}.$$

Clearly $\Omega(\mathcal{U}) \subseteq [1,\infty)$. It is an important problem to describe the precise structure of the set $\Omega(\mathcal{U})$ for various classes $\mathcal{U}$. The main question we are interested here is whether 1 is an accumulation point of $\Omega(\mathcal{U})$. Certain considerations of such a kind can be found in [18], where Grigorchuk and de la Harpe showed that there exist finitely generated groups of exponential growth with growth rates arbitrary close to 1. However, there are numerous questions, which are still open. Probably the most interesting one is

Question 2.1. *Let $\mathcal{H}$ denotes the set of all hyperbolic groups which are non-elementary (i.e., not cyclic–by–finite). Is 1 an accumulation point of $\Omega(\mathcal{U})$?*

It was pointed out in [33], that the positive answer would imply the existence of a relatively simple construction of a finitely generated group of non–uniform exponential growth. We note that the groups constructed in [18] are not hyperbolic, but they are non–amenable and semihyperbolic in the sense of [2].

We obtain the following analog of the Grigorchuk – de la Harpe result in the case of elementary amenable groups.

Theorem 2.3 *The number* 1 *is an accumulation point of the set* $\Omega(EG)$.

In contrast, we provide one result of the converse type.

Theorem 2.4. *Let* $\mathcal{M}$ *denote the set of all finitely generated metabelian non–polycyclic groups. Then there is* $\varepsilon > 0$ *such that* $\omega(G) > 1 + \varepsilon$ *for any* $G \in \mathcal{M}$.

It is worth to mention that we do not know whether the analog of Theorem 2.4 is true in the case of polycyclic groups.

3 Outline of the proof of main theorems.

Here we describe shortly the main idea of the proof of Theorems 2.1 and 2.2. Recall that a group P is called *polycyclic* if there is a finite subnormal series

$$1 = P_k \lhd P_{k-1} \lhd \ldots \lhd P_0 = P, \tag{2}$$

where P_{i-1}/P_i is cyclic for every $i = 1, \ldots, k$. We begin by proving the following 'dichotomy'.

Proposition 3.1. *Let* $G \in EG$ *be a finitely generated group. Suppose, in addition, that* G *is not virtually nilpotent. Then there exists a normal subgroup* H *of* G *such that* G/H *is virtually polycyclic and at least one of the following conditions holds.*

1) G/H *has exponential growth.*

2) H *is not finitely generated.*

Thus the proof of Theorem 2.1 is divided into two parts depending on the existence of a finite generating set of the subgroup H from Proposition 3.1. The first case is relatively simple, however we need two auxiliary results to treat it.

Lemma 3.2. *Let* G *be a finitely generated group. Then the following assertions are true.*

1) Suppose R *is a normal subgroup of* G*; then* $\omega(G/R) \leq \omega(G)$.

2) Suppose R *is a subgroup of finite index in* G*; then* $\omega(R) \leq \omega(G)^{(2[G:R]-1)}$.

The proof of claim *1)* is straightforward and is left as an exercise. Claim *2)* follows, for example, from Proposition 3.3 of [37] (and can also be proved by the reader by using straightforward arguments).

Theorem 3.1. [32, Theorem 1.1.] *Let* G *be a finitely generated solvable group of zero entropy. Then* G *contains a nilpotent subgroup of finite index.*

By combining the second assertion of Lemma 3.1 and Theorem 3.1, we obtain that any virtually polycyclic group of exponential growth is of uniform

exponential growth, since every finitely generated virtually nilpotent group has polynomial growth [7]. Further, if a group G has a quotient group of uniform exponential growth, then it is of uniform exponential growth itself by the first assertion of Lemma 3.1. This shows that G has uniform exponential growth in case the condition 1) of Proposition 3.1 holds.

The second case is more complicated. Without loss of generality, we can assume the quotient group G/H to be not of exponential growth, i.e., to be virtually nilpotent. Moreover, since the property to be of uniform exponential growth is preserved under the taking of subgroups of finite index, we can assume G/H to be nilpotent. In this settings the following proposition plays the crucial role in our proof.

Proposition 3.2. *Let G be a finitely generated group such that there exists an exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow N \longrightarrow 1,$$

where N is nilpotent of degree d and K is not finitely generated. Then we have

$$\omega(G) \geq \sqrt[\alpha]{2}, \tag{3}$$

where $\alpha = 3 \cdot 4^{d+1}$.

Now let us turn to Theorem 2.2. The conditions of the theorem together with Theorem 2.1 and the first assertion of Lemma 4.1 imply that G/H is virtually nilpotent. Therefore, G contains a subgroup G_0 of finite index such that $H \lhd G_0$ and G_0/H is nilpotent. By the second assertion of Lemma 3.1, we have $\omega(G_0) = 1$. Therefore, H is finitely generated by Proposition 3.2

Thus to complete the proofs of Theorems 2.1 and 2.2 it remains to prove Propositions 3.1 and 3.2. The first proposition is obtained in the next section. The proof of the second one involves methods based on commutator calculus. We give this proof in Section 6 modulo some auxiliary results, which are obtained in Section 5.

4 Description and some properties of elementary amenable groups.

First we recall the description of elementary classes of groups given in [33]. In case of elementary amenable groups this description is slightly stronger than the Chou one [10].

Definition 4.1. Let B be a class of groups. The *elementary class of groups with the base* B is the smallest class of groups which contains B and is closed under operations (S)–(U).

Now we fix B. Let $\mathcal{E}_0(B)$ consist of the trivial group only. Assume that $\alpha > 0$ is an ordinal and that we have defined $\mathcal{E}_\beta(B)$ for each ordinal $\beta < \alpha$. If

α is a limit ordinal, set

$$\mathcal{E}_\alpha(B) = \bigcup_{\beta<\alpha} \mathcal{E}_\beta(B),$$

and if α is successor, let $\mathcal{E}_\alpha(B)$ be the class of groups that can be obtained from groups of $\mathcal{E}_{\alpha-1}(B)$ by applying operation (U) or the following operation once.

(E_0) Given a group, take its extension by a group from B.

Lemma 4.1. [33] *The class $\mathcal{E}_\alpha$ is closed under operation (S) and (Q) (SQ–closed for brevity) for each ordinal α.*

Theorem 4.1. [33] *Let B be a class of groups. Assume that B is closed under the operations (S) and (Q). Then we have*

$$\mathcal{E}(B) = \bigcup_{\alpha} \mathcal{E}_\alpha(B),$$

where the union is taken over all ordinal numbers.

Example 4.1. Let us take $B = A \cup F$, where A and F are the classes of all abelian and finite groups respectively. Then the corresponding elementary class is precisely EG, as follows from Theorem 4.1.

The next lemma is well–known (see for example [6] or [36]).

Lemma 4.2. *Any extension of a virtually polycyclic group by a virtually polycyclic group is a virtually polycyclic group.*

Now we are ready to formulate the main result of this section. Instead of Proposition 3.1, we will prove a stronger result by transfinite induction on α. Recall that a subgroup H of a group G is called characteristic if for any automorphism ϕ of G, one has $\phi(H) \leq H$. Evidently if G is a normal subgroup of a group F and H is a characteristic subgroup of G, then H is normal in F.

Lemma 4.3. *Let $G \in EG_\alpha$ be a finitely generated group. Suppose, in addition, that G is not virtually nilpotent. Then there exists a characteristic subgroup H of G such that G/H is virtually polycyclic and at least one of the following conditions holds.*

1) G/H has exponential growth.

2) H is not finitely generated.

Proof. The case $\alpha = 0$ is trivial. Suppose that $\alpha > 0$. First assume that α is a limit ordinal. Then $G \in EG_\beta$ for some $\beta < \alpha$ and thus the assertion of the lemma holds by the inductive assumption.

Now let α be a non–limit ordinal. Assume that $G = \bigcup\limits_{\lambda\in\Lambda} G_\lambda$, where $G_\lambda \in EG_{\alpha-1}$. Since G is finitely generated, we have $G = G_{\lambda_0}$ for some λ_0. Hence the assertion of the lemma is true by the inductive hypothesis again. Further, suppose G is an extension of the form

$$1 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 1,$$

where $M \in EG_{\alpha-1}$ and L is abelian or finite. First we consider the case of abelian L. Take $G' = [G, G]$ and observe that $G' \leq M$ and thus $G' \in EG_{\alpha-1}$ by Lemma 4.1. If G' is not finitely generated, we can take it for H. Otherwise, there are two possibilities. The first one is the case of virtually nilpotent G'. Clearly, then G is virtually polycyclic by Lemma 4.2. Taking into account that G is not virtually nilpotent, we conclude that G is of exponential growth.

Now suppose that G' is not virtually nilpotent. Then, by the inductive hypothesis, there exists a characteristic subgroup $H \leq G'$ satisfying the requirements of the proposition. Clearly, H is a characteristic subgroup of G. It remains to notice that since G'/H is virtually polycyclic, then G/H is virtually polycyclic. Moreover, since G'/H is of exponential growth, so is G/H.

Similarly, if L is finite, say $|L| = m$, then we take the subgroup $G^m = \langle g^m : g \in G\rangle$. Evidently $G^m \leq M$ and the rest of the proof is essentially the same as in the previous case. Additionally we only need the fact that any finitely generated periodic group from EG is finite (see [10]). $\square$

5 Technical lemmas.

Throughout this section we fix a group G generated by a finite set X and denote by $\mathcal{L}(X \cup X^{-1})$ the set of all words over $X \cup X^{-1}$. For two words $u, v \in \mathcal{L}(X \cup X^{-1})$, we write $u \equiv v$ to express the letter–for–letter equality, and $u = v$ if u and v represent the same element of G. By $\|w\|$ we denote the length of a word $w \in \mathcal{L}(X \cup X^{-1})$. We write also u^v instead of $v^{-1}uv$ and $[u, v]$ instead of $u^{-1}v^{-1}uv$.

We fix arbitrary finite subsets $V, W \in \mathcal{L}(X \cup X^{-1})$ and assume in addition that V and W satisfy the following conditions.

(I) $X^{\pm 1} \in V$.

(II) The set V is ordered, i.e., $V = \{v_1, v_2, \ldots, v_p\}$. Moreover, set $V_i = \{v_1, v_2, \ldots, v_i\}$ for each $i = 1, \ldots p$; then either

$$[V_i, V_j] \subseteq V_{\min\{i,j\}-1}$$

or

$$[V_i, V_j] \subseteq W$$

for any $i, j = 1, \ldots, p$.

(III) For any $i = 1, \ldots, p$, and any $w \in W$, the normal closure

$$\langle w\rangle^{\langle v_i\rangle} = \langle v_i^{-l} w v_i^l \; : \; l \in \mathbb{Z}\rangle$$

is finitely generated, i.e., there exists $L_i \in \mathbb{N}$ such that

$$\langle w\rangle^{\langle v_i\rangle} = \langle v_i^{-l} w v_i^l \; : \; |l| \leq L_i\rangle.$$

We are going to show that $\langle W\rangle^G$ is finitely generated as a subgroup. To reach this this goal we need one more auxiliary definition. Let v be a word in the alphabet $V^{\pm 1} \cup W^{\pm 1}$. Denote by $\lambda_i(v)$ the number of appearances of the letters $v_i^{\pm 1}$ in v. For instance, if $v \equiv v_1 v_2 v_1^{-1}$, then $\lambda_1(v) = 2$, $\lambda_2(v) = 1$. We note also that λ_i is defined just for words over $V^{\pm 1} \cup W^{\pm 1}$, not for elements of G, as different words can represent the same element.

Given X, G, V, and W as described above, we set

$$L = \max_{i=1,\dots,p} L_i$$

and

$$Z = \{v^{-1} w v \ : \ w \in W,\ v \in \mathcal{L}(V^{\pm 1}),\ \lambda_i(v) \le L \ \forall i = 1, \dots, p\}. \tag{4}$$

We will say that the above decomposition $z \equiv v^{-1} w v$ of a word $z \in Z$, where $w \in W,\ v \in \mathcal{L}(V^{\pm 1})$, is a *canonical form* of an element $z \in Z$; clearly $\lambda_i(z) = 2\lambda_i(v)$.

The main result of this section is the following.

Lemma 5.1. *In the above notation, we have $\langle W\rangle^G = \langle Z\rangle$, i.e., the normal closure of W in the group G is generated by the set Z as a subgroup.*

The proof consists of four lemmas. Denote by $(x, y^{\pm 1})_1$ the set $\{[x, y], [x, y^{-1}]\}$. Further, we define

$$(x, y^{\pm 1})_{i+1} = \{[c, y], [c, y^{-1}] \ : \ c \in (x, y^{\pm 1})_i\}.$$

Lemma 5.2. *Let H be a group, $a, b \in H$. Then*

$$(a, b^{\pm 1})_n \subseteq \langle a^{b^l} \ : \ l = -n, \dots, n\}$$

for any $n \in \mathbb{N}$.

Proof. For $n = 1$, we have

$$(a, b^{\pm 1})_1 = \{a^{-1} a^b, a^{-1} a^{b^{-1}}\} \subseteq \langle a, a^{b^{\pm 1}}\rangle.$$

Now suppose $n > 1$. By induction, we can assume that the assertion of the lemma is true for $(n-1)$, i.e.,

$$(a, b^{\pm 1})_{n-1} \subseteq \langle a^{b^l} \ : \ l = -n+1, \dots, n-1\rangle. \tag{5}$$

Denote $a_l = a^{b^l}$ for brevity and consider an element

$$c = a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m} \in (a, b^{\pm 1})_{n-1}.$$

By (5), we can assume that $|l_j| \le n - 1$ for any $j = 1, \dots, m$. We obtain

$$[c, b] \equiv c^{-1} c^b = (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^{-1} (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^b = (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^{-1} (a_{l_1+1}^{\alpha_1} \dots a_{l_m+1}^{\alpha_m}).$$

Therefore, $[c,b] \in \langle a_{-n+1},\ldots,a_n\rangle$. Similarly we obtain $[c,b^{-1}] \in \langle a_{-n},\ldots,a_{n-1}\rangle$. The lemma is proved. $\square$

The following three lemmas will be proved by common induction on a parameter r.

Lemma 5.3. *Let $0 \le n_1 < n_2 < \ldots < n_m$ be a sequence of integers. Consider a word*

$$\bar{v} \equiv (a_1 \ldots a_{n_1} v_r^{\epsilon_1}) \cdot (a_{n_1+1} \ldots a_{n_2} v_r^{\epsilon_2}) \cdot \ldots \cdot (a_{n_{m-1}+1} \ldots a_{n_m} v_r^{\epsilon_m}), \tag{6}$$

where $a_i \in (V \setminus \{v_r\})^{\pm 1}$, $\epsilon_i \in \mathbb{Z}$ for each $i = 1,\ldots,m$, and

$$\sum_{i=1}^{m} |\epsilon_i| \le L+1. \tag{7}$$

Then we have

$$\bar{v} = v_r^{\sigma} \cdot a_1 b_1 \cdot a_2 b_2 \cdot \ldots \cdot a_{n_m} b_{n_m}, \tag{8}$$

where $\sigma = \sum\limits_{i=1}^{m} \epsilon_i$ and

$$b_i \in \left\langle V_{r-1} \cup \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right) \right\rangle \tag{9}$$

for all i. In particular, $b_i \in \langle V_{r-1} \cup Z\rangle$.

Lemma 5.4. *Suppose that $z \equiv v^{-1}wv$ is a canonical form of an element $z \in Z$ and b is a word over $(V \setminus \{v_r\})^{\pm 1} \cup \left(\bigcup\limits_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$. Then we have*

$$z^b \in \left\langle \begin{array}{ccl} & & y_0 \in \mathcal{L}(V^{\pm 1}),\ w_0 \in W, \\ y_0^{-1} w_0 y_0 & : & \lambda_i(y_0) \le L \ \forall\, i = 1,\ldots,r, \\ & & \lambda_i(y_0) \le \lambda_i(v) + \lambda_i(b)\ \forall\, i = r+1,\ldots,p \end{array} \right\rangle. \tag{10}$$

In particular, if

$$\lambda_i(v) + \lambda_i(b) \le L \tag{11}$$

for each $i = r+1,\ldots,p$, then $z^b \in \langle Z\rangle$.

Lemma 5.5. *For any $z = v^{-1}wv \in Z$ and any $t \in (V_r)^{\pm 1}$, we have*

$$z^t \in \left\langle \begin{array}{ccl} & & y_0 \in \mathcal{L}(V^{\pm 1}),\ w_0 \in W, \\ y_0^{-1} w_0 y_0 & : & \lambda_i(y_0) \le L \ \forall\, i = 1,\ldots,r, \\ & & \lambda_i(y_0) \le \lambda_i(v) + \lambda_i(t)\ \forall\, i = r+1,\ldots,p \end{array} \right\rangle. \tag{12}$$

In particular, $z^t \in \langle Z\rangle$.

Proof. For all lemmas the case $r = 1$ is essentially the same as the inductive step. Thus we assume Lemmas 5.3 – 5.5 to be true for all positive integers $q < r$ whenever $r > 1$ (and assume nothing if $r = 1$).

Proof of Lemma 5.3. Using the formula

$$xy = yx[x, y]$$

we can collect all appearances of the letter $v_r^{\pm 1}$ in the word $\bar{v}$ from right to left in order to obtain a word of the form (8). It is easy to check that each b_i will be a product of elements of the sets $(a_i, v_r^{\pm 1})_n$, where

$$n \leq \sum_{i=1}^{m} |\epsilon_i| \tag{13}$$

For an element $u \in (a_i, v_r^{\pm 1})_n$, there are two possibilities.

(a) First assume that $u \in V$. Then $u \in V_{r-1}$ by condition (II) (see the beginning of the section).

(b) Suppose $u \notin V$. Consider the minimal n_0 such that $(a_i, v_r^{\pm 1})_{n_0} \nsubseteq V$. Clearly, $a_i \in V^{\pm 1}$ implies that $n_0 \geq 1$. By Lemma A.2,

$$u \in \big((a_i, v_r^{\pm 1})_{n_0}, v_r^{\pm 1}\big)_{n-n_0} \subseteq \left\langle \big((a_i, v_r^{\pm 1})_{n_0}\big)^{v_r^l} \;:\; |l| \leq n - n_0 \right\rangle.$$

Using (7) and (13), we note that

$$n - n_0 \leq \sum_{i=1}^{m} |\epsilon_i| - n_0 \leq L$$

and hence

$$u \in \left\langle \big((a_i, v_r^{\pm 1})_{n_0}\big)^{v_r^l} \;:\; |l| \leq L \right\rangle \leq \left\langle \bigcup_{j=-L}^{L} W^{v_r^j} \right\rangle.$$

Indeed, by minimality of n_0, we have $(a_i, v_r^{\pm 1})_{n_0 - 1} \in V$. Now using condition (II), we obtain $(a_i, v_r^{\pm 1})_{n_0} = \big((a_i, v_r^{\pm 1})_{n_0-1}, v_r^{\pm 1}\big) \subseteq W$

Thus in both cases

$$u \in \left\langle \bigcup_{j=-L}^{L} W^{v_r^j} \cup V_{r-1} \right\rangle,$$

and, therefore, the same is true for each b_i. The lemma is proved.

Proof of Lemma 5.4. Denote by Y the group situating at the right side of (10). The proof will be by induction on the length of the word b. The case $|b| = 0$ is trivial. Now suppose $|b| = n + 1 \geq 1$. Then $b = a_0 a_1$, where $a_1 \in (V \setminus \{v_r\})^{\pm 1} \cup \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$ and a_0 has length n. By the inductive assumption, we have

$$z^b = z^{a_0 a_1} = (z_1 \ldots z_q)^{a_1} = z_1^{a_1} \ldots z_q^{a_1}, \tag{14}$$

where $z_j = y_j^{-1} w_j y_j$ are some elements such that $w_j \in W^{\pm 1}$, $y_j \in \mathcal{L}(V^{\pm 1})$, $\lambda_i(y_j) \leq \lambda_i(a_0) + \lambda_i(v)$ for all $i = r+1, \ldots, p$, and $\lambda_i(y_j) \leq L$ for all $i = 1, \ldots, r$.

Now let us consider $z_j^{a_1}$ for some j and prove that $z_j^{a_1} \in Y$. There are three possibilities.

(a) $a_1 \in \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$. Evidently $a_1 \in Z$ in this case. Moreover, $a_1 \in Y$ and hence $z_j^{a_1} \in Y$.

(b) $a_1 \in (V_{r-1})^{\pm 1}$. We note that this case is impossible if $r = 1$. If $r > 1$, we assume that Lemma 5.5 has already been proved for all smaller volumes of the parameter. Thus we obtain $z_j^{a_1} \in Y$ applying Lemma 5.5 for $t \equiv a_1$, $z \equiv z_j$.

(c) $a_1 \in (V \setminus V_r)^{\pm 1}$. Suppose $a_1 \equiv v_k$ for some $k \in \{r+1, \ldots, p\}$. For the element $z_j^{a_1}$ consider its canonical form, the word $v_k^{-1} y_j^{-1} w_j y_j v_k$, obtained from the canonical form of the element z_j. We have

$$\lambda_k(v_k y_j) = \lambda_k(y_j) + 1 \leq \lambda_k(a_0) + \lambda_k(v) + 1 = \lambda_k(b) + \lambda_k(v).$$

Clearly, if $i \neq k$, then $\lambda_i(v_k y_j) = \lambda_i(y_j)$. This shows that $z_j^{a_1}$ lies in Y again.

Since $z_j^{a_1} \in Y$ is true for each factor of type $z_j^{a_1}$ in (14), we obtain $z^b \in Y$ and the proof of the lemma is completed.

Proof of Lemma 5.5. Denote by F the group at the right side of (12). In view of inductive arguments, it is sufficient to consider the case $t \equiv v_r^{\pm 1}$. Assume that $t \equiv v_r$ for convenience (the case $t \equiv v_r^{-1}$ is analogous). First suppose that $\lambda_r(v) \leq L - 1$, i.e., $\lambda_r(z) \leq 2L - 2$. Note that

$$\lambda_i(z^{v_r}) = \begin{cases} \lambda_i(z), \text{ if } i \neq r, \\ \lambda_i(z) + 2, \text{ if } i = r. \end{cases}$$

Thus $\lambda_i(z^{v_r}) \leq 2L$ for $i = 1, \ldots, r$, and $\lambda_i(z^{v_r}) = \lambda_i(z)$ for $i = r+1, \ldots, p$. This means that $z^{v_r} \in F$.

Now let $\lambda_r(v) = L$. Then $\lambda_r(v v_r) = L + 1$ and the word $\bar{v} \equiv v v_r$ has the form (6). Applying Lemma 5.3, we obtain

$$\bar{v} = v_r^{\sigma} b,$$

where

$$|\sigma| = \left| \sum_{i=1}^{m} \epsilon_i \right| \leq \lambda_r(\bar{v}) = L + 1$$

and b satisfies the condition $\lambda_i(b) = \lambda_i(v) \leq L$ for all $i = r+1, \ldots, p$ (obviously this condition follows from (9)). In case $|\sigma| \leq L$ we do nothing. If $|\sigma| = L + 1$, we apply condition (III) and obtain

$$z^{v_r} = w^{v_r^{\sigma} b} = \left(\prod_{i=-L}^{L} \left(w^{v_r^i} \right)^{\xi_i} \right)^b = \prod_{i=-L}^{L} \left(\left(w^{v_r^i} \right)^b \right)^{\xi_i}. \tag{15}$$

Finally, we consider the elements $(w^{v_r^j})^b$, where $|j| \leq L$. In order to finish the proof of Lemma 5.5 it remains to show that these elements belong to F. The element b satisfies the conditions of Lemma 5.4, as it contains no appearances of the letters $v_r^{\pm 1}$. Thus $\left(w^{v_r^i}\right)^b \in F$ by Lemma 5.4. It follows that $z^{v_r} \in F$. The same arguments show that $z^{v_r^{-1}} \in F$. The lemma is proved and the inductive step is completed. □

Proof of Lemma 5.1. Lemma 5.5 implies that $z^t \in \langle Z \rangle$ for any $z \in Z$, $t \in V$. Since $X^{\pm 1} \subseteq V$, we have $z^g \in \langle Z \rangle$ for any $g \in G$. This means that $\langle Z \rangle^G = \langle Z \rangle$ and the lemma is proved. □

6 Estimating exponential growth rates from below

For any group H, let $\gamma_i H$ be the i-th term of the lower central series

$$H = \gamma_1 H \rhd \gamma_2 H \rhd \dots,$$

where $\gamma_{i+1} H = [\gamma_i H, H]$. Recall that a group N is called *nilpotent of degree* t if $\gamma_{t+1} N = 1$. Finally, given subsets $Y, Z \subseteq G$, let $\langle Y \rangle$ denote the subgroup generated by Y, and $\langle Y \rangle^Z$ the subgroup generated by all elements of type $z^{-1} y z$, where $y \in Y, z \in Z$. Thus $\langle Y \rangle^G$ is the normal closure of Y in G.

Definition 6.1. Let G be a group with a given finite generating set X. For any finite subset $Y = \{y_1, \dots, y_m\} \subseteq G$, we define its *depth* with respect to X as follows

$$depth_X(Y) = \max_{i=1,\dots,m} ||y_i||_X.$$

If H is a finitely generated subgroup of G, then we define its depth with respect to X by setting

$$depth_X(H) = \min_{H=\langle Y \rangle} depth_X(Y),$$

where the minimum is taken over all finite generating sets of H.

Lemma 6.1. *Suppose that G is a group with a given finite generating set X and R is a finitely generated subgroup of G; then we have*

$$\omega(G, X) \geq (\omega(R))^{\frac{1}{depth_X(R)}}.$$

The proof is straightforward and is left as an exercise to the reader. □

Let us introduce certain auxiliary notation. As above, suppose G is a group generated by a finite set X. Then we set $W_1(X) = X \cup X^{-1}$ and, by induction,

$$W_i(X) = \{[u^{\pm 1}, v^{\pm 1}] \ : \ u \in W(i_1), v \in W(i_2), i_1, i_2 \in \mathbb{N}, i_1 + i_2 = i\}$$

for any $i > 1$. We write *weight* $(v) = i$ for a word $v \in \mathcal{L}(X \cup X^{-1})$, if $v \in W_i$. Also, consider the function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(1) = 1 \text{ and } f(n+1) = 2f(n) + 2 \tag{16}$$

for any $n \in \mathbb{N}$. It can easily be checked that $f(n) = 3 \cdot 2^{n-1} - 2$. The following lemma is quite trivial.

Lemma 6.2. *Let f be the function given by (16). Then for any $i, j \in \mathbb{N}$, one has*

$$2(f(i) + f(j)) \leq f(i+j).$$

Lemma 6.3. *For any group G with a given finite generating set X, one has*

$$depth_X(W_n(X)) \leq f(n). \tag{17}$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Further, for $n > 1$, we observe that if $u \in W_{i_1}(X), v \in W_{i_2}(X)$ and $i_1 + i_2 = n$, then

$$\begin{aligned} ||[u^{\pm 1}, v^{\pm 1}]||_X \leq \; & 2(||u||_X + ||v||_X) \leq 2(depth_X(W_{i_1}(X)) + depth_X(W_{i_2}(X))) \\ & \leq 2(f(i_1) + f(i_2)) \leq f(n) \end{aligned}$$

by the inductive hypothesis and Lemma 6.2. □

As an exercise, one can show that if G is a non abelian free group and X is a basis in G, then $depth_X(W_n(X)) = f(n)$.

The next lemma follows from the results of the previous section.

Lemma 6.4. *Suppose that G is a finitely generated group and, for some $s \in \mathbb{N}$, $s \geq 2$, all subgroups of type*

$$H_{v,w} = \langle v^{-l} w v^l \ : \ l \in \mathbb{Z} \rangle \tag{18}$$

are finitely generated for any $v \in \bigcup_{j=1}^{s-1} W_j$, $w \in \bigcup_{j=s}^{2s} W_j$. Then $\gamma_s(G)$ is finitely generated .

Proof. The reader can easily check that the sets

$$W = \bigcup_{j=s}^{2s} W_j$$

and

$$V = \bigcup_{j=1}^{s-1} W_j$$

satisfy hypotheses (I) – (III) listed at the beginning of Section 5. Indeed, (I) is obvious. To satisfy (II), we just need to order commutators in V in such a way that $weight(v_i) \leq weight(v_j)$ whenever $i \geq j$. Finally, (III) follows from the conditions of Lemma 6.4. It remains to note that $\gamma_s(G) = \langle W \rangle^G$, as $\gamma_s(G)$ is generated by $\bigcup_{t \geq s} W_t$ (see [25, Ch.5]) and $W_t \in \langle W \rangle^G$ for every $t > 2s$ by the definition of W_t. $\Box$

Proof of Proposition 3.2. Let X be some finite generating set of G. Let us put $s = d + 1$. We would like to show that there is a subgroup $H_{v,w} \leq G$ of type (18) having no finite set of generators. Indeed, suppose that all $H_{v,w}$ are finitely generated. Then $\gamma_s G$ is finitely generated by Lemma 6.4. Clearly $\gamma_s G \lhd K$. Therefore, $K/\gamma_s G$ is a subgroup of a finitely generated nilpotent group $G/\gamma_s G$ and thus is finitely generated. It follows that K is finitely generated and we arrive at contradiction.

Thus there exists $H_{v,w}$ which is infinitely generated. Consider the subgroup $H = \langle v, w \rangle$. For any sequence $\alpha = (\alpha_1, \ldots, \alpha_p)$, $p \in \mathbb{N}$, where $\alpha_i \in \{0, 1\}$ for each $i = 1, \ldots p$, we define an element $t(\alpha)$ by the formula

$$t(\alpha) = w^{\alpha_1} v w^{\alpha_2} v \ldots w^{\alpha_p} v.$$

Suppose $t(\alpha) = t(\beta)$ for some $\alpha = (\alpha_1, \ldots, \alpha_p) \neq (\beta_1, \ldots, \beta_q) = \beta$. Notice that $H_{v,w}$ is normal in H and $H/H_{v,w}$ is cyclic. Furthermore, $H/H_{v,w}$ is infinite. Indeed, otherwise $H_{v,w}$ is finitely generated). Hence $vH_{v,w}$ has infinite order when regarded as an element of $H/H_{v,w}$. This implies $p = q$ and we have

$$w^{\alpha_1} v w^{\alpha_2} v \ldots w^{\alpha_p} v = w^{\beta_1} v w^{\beta_2} v \ldots w^{\beta_p} v. \tag{19}$$

Without loss of generality, we can assume $\alpha_1 \neq \beta_1$ and $\alpha_p \neq \beta_p$. Denote by w_l the element w^{v^l}. Then (19) can be rewritten as

$$(w_p)^{\alpha_1} (w_{p-1})^{\alpha_2} \ldots (w_1)^{\alpha_p} = (w_p)^{\beta_1} (w_{p-1})^{\beta_2} \ldots (w_1)^{\beta_p},$$

or, equivalently,

$$(w_p)^{\alpha_1 - \beta_1} = (w_{p-1})^{\beta_2} \ldots (w_1)^{\beta_p} \left((w_{p-1})^{\alpha_2} \ldots (w_1)^{\alpha_p} \right)^{-1}.$$

Note that $\alpha_1 - \beta_1 = \pm 1$. Therefore,

$$w_p \in \langle w_1, \ldots, w_{p-1} \rangle. \tag{20}$$

Conjugating by v and using (20), we obtain

$$w_{p+1} = w_p^v \in \langle w_2, \ldots, w_p \rangle \leq \langle w_1, \ldots, w_{p-1} \rangle$$

and so on. By induction, $w_n \in \langle w_1, \ldots, w_{p-1} \rangle$ for any $n \geq p$. Similarly, we can show that $w_n \in \langle w_2, \ldots, w_p \rangle$ for any $n \leq 1$. Hence $w_n \in \langle w_1, \ldots, w_p \rangle$ for any $n \in \mathbb{Z}$ that contradicts to the assumption that $H_{v,w}$ is infinitely generated.

This shows that $t(\alpha) \neq t(\beta)$ whenever $\alpha \neq \beta$. Recall that $||v||_X \leq f(s-1)$ and $||w||_X \leq f(2s)$ by Lemma 6.3. Hence we have

$$||w^{\alpha_1} v w^{\alpha_2} v \dots w^{\alpha_p} v||_X \leq p(||w||_X + ||v||_X) \leq p(f(s-1) + f(2s)) \leq 2pf(2s).$$

Thus,

$$\begin{array}{rl} \gamma_H^X(n) \geq & card\,\{t(\alpha)\ :\ ||t(\alpha)||_X \leq n\} \\ \geq & card\,\{(\alpha_1, \dots, \alpha_p)\ :\ \alpha_1, \dots, \alpha_p \in \{0,1\},\ p \leq c(n)\} = 2^{c(n)}, \end{array}$$

where

$$c(n) = \left[\frac{n}{2f(2s)}\right].$$

Here $[x]$ means the integral part of x. This implies

$$\omega(H, X) \geq \sqrt[\beta]{2} \tag{21}$$

for $\beta = 2f(2s)$. Note that $2f(2s) = 2(3\cdot 2^{2s-1} - 2) \leq 6\cdot 2^{2s-1} = 6\cdot 2^{2d+1} = 12\cdot 4^d$. Since (21) is true for an arbitrary generating set X, we obtain (3). $\square$

7 Applications

In this section we consider two applications of the results obtained above. Our goal is to prove Theorems 2.3 and 2.4.

Definition 7.1. The *Cayley graph* $\Gamma = \Gamma(G, S)$ of a group G generated by a set S, is an oriented labeled 1–complex with the vertex set $V(\Gamma) = G$ and the edge set $E(\Gamma) = G \times S$. An edge $e = (g, s) \in E(\Gamma)$ goes from the vertex g to the vertex gs and has the label $\phi(e) = s$. As usual, we denote the origin and the terminus of the edge e, i.e., the vertices g and gs, by $\alpha(e)$ and $\omega(e)$ respectively. One can endow the group G (and, therefore, the vertex set of Γ) with a *length function* by assuming $\|g\|_S$, the length of an element $g \in G$, to be equal to the length of a shortest word in the alphabet $S \cup S^{-1}$ representing g.

Definition 7.2. Two groups G and H with the same finite sets of generators are called n–isomorphic if their Cayley graphs restricted to balls of radius n centered at the identity are isomorphic (as labeled oriented 1–complexes). This notion gives rise to the topology of local isomorphism (the Grigorchuk's topology) on the set of all groups having the same collection of generators, with subsets of n–isomorphic groups as the base of neighborhoods. The local topology has interesting applications in group theory (see [38], [39], [16], [34]).

Lemma 7.1. *Suppose that the group G is a limit of groups G_i, $i \in \mathbb{N}$, with respect to the Grigorchuk topology (in this case we say that G is approximated by G_i). Let X denote the common generating set of G and G_i's. Then*

$$\lim_{i\to\infty} \omega(G_i, X) \leq \omega(G, X).$$

Proof. In case the group G is of intermediate growth, the proof can be found in [19]. We reproduce it here with small changes. It is well known that

$$\omega(G, X) = \inf_{n} \sqrt[n]{\gamma_G^X(n)}$$

(see, for example, [18]). For any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\sqrt[m]{\gamma_G^X(m)} < \omega(G, X) + \varepsilon.$$

Further, by the definition of the local topology, there exists $N \in \mathbb{N}$ such that for every $i > N$, the balls of radius m centered at the identity in the groups G and G_i coincide. In particular, this implies

$$\gamma_{G_i}^X(m) = \gamma_G^X(m).$$

Finally, we have

$$\omega(G_i, X) = \inf_{n} \sqrt[n]{\gamma_{G_i}^X(n)} \leq \sqrt[m]{\gamma_{G_i}^X(m)} = \sqrt[m]{\gamma_G^X(m)} < \omega(G, X) + \varepsilon. \tag{22}$$

Since we obtain (22) for an arbitrary $\varepsilon > 0$, the lemma is proved. □

Proof of Theorem 2.3. In [16], Grigorchuk constructed a set of 3–generated groups, $\mathcal{X}$, which contains a number of groups of subexponential growth and a subset $\mathcal{S}$ of elementary amenable (moreover, virtually solvable) groups of exponential growth such that $\mathcal{S}$ is dense in $\mathcal{X}$. Thus there exists a finitely generated group G of subexponential growth generated by a finite set X and a sequence of elementary amenable groups G_i of exponential growth (generated by the same set) that converges to G. By Lemma 7.1, we have

$$\lim_{i\to\infty} \omega(G_i) \leq \lim_{i\to\infty} \omega(G_i, X) \leq \omega(G, X) = 1.$$

□

Proof of Theorem 2.4. Let G be a finitely generated metabelian group. If the derived subgroup $[G, G]$ is finitely generated, then G is polycyclic. Thus we can apply Proposition 3.3 for $K = [G, G]$. In this case the group $N = G/[G, G]$ is abelian. Substituting $d = 1$ in (3), we obtain

$$\omega(G) \geq \sqrt[48]{2}.$$

□

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